MATH 245 S21, Exam 2 Solutions

2. Prove that $\forall x \in \mathbb{R}, \exists! y \in \mathbb{R},\ (x = \lfloor x \rfloor + y) \wedge (0 \leq y < 1)$.

EXISTENCE: Let $x \in \mathbb{R}$ be arbitrary, and choose $y = x - \lfloor x \rfloor$. This choice of y forces $x = \lfloor x \rfloor + y$. Now (by definition of floor), $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$. Subtracting $\lfloor x \rfloor$ throughout gives $0 \leq x - \lfloor x \rfloor < 1$, hence $0 \leq y < 1$. By addition, $(x = \lfloor x \rfloor + y) \wedge (0 \leq y < 1)$.

UNIQUENESS: Let $x \in \mathbb{R}$ be arbitrary, and suppose that there are $y, z \in \mathbb{R}$ with $x = \lfloor x \rfloor + y, 0 \leq y < 1, x = \lfloor x \rfloor + z, 0 \leq z < 1$. We combine the first and third of these to get $\lfloor x \rfloor + y = \lfloor x \rfloor + z$. Subtracting $\lfloor x \rfloor$ from both sides, we get $y = z$.

3. Use the division algorithm to prove that $\forall n \in \mathbb{N},\ \frac{n^2+9n+20}{2} \in \mathbb{Z}$.

Let $n \in \mathbb{N}$ be arbitrary. We apply the DA to $n, 2$ to get unique integers q, r with $n = 2q + r$ and $0 \leq r < 2$. The conditions on r allow only two possibilities: $r = 0, 1$.

Case $r = 0$: Now $n = 2q$, so we substitute and get $\frac{n^2+9n+20}{2} = \frac{(2q)^2+9(2q)+20}{2} = \frac{4q^2+18q+20}{2} = 2q^2+9q+10 \in \mathbb{Z}$.

Case $r = 1$: Now $n = 2q+1$, so we substitute and get $\frac{n^2+9n+20}{2} = \frac{(2q+1)^2+9(2q+1)+20}{2} = \frac{4q^2+4q+1+18q+9+20}{2} = \frac{4q^2+22q+30}{2} = 2q^2 + 11q + 15 \in \mathbb{Z}$.

In both cases, $\frac{n^2+9n+20}{2} \in \mathbb{Z}$.

4. Use (some form of) mathematical induction to prove that $\forall n \in \mathbb{N},\ \frac{n^2+9n+20}{2} \in \mathbb{Z}$.

We can use vanilla induction. Base case, $n = 1$, we have $\frac{n^2+9n+20}{2} = \frac{1+9+20}{2} = 15 \in \mathbb{Z}$.

Inductive case: Let $n \in \mathbb{N}$ be arbitrary and assume that $\frac{n^2+9n+20}{2} \in \mathbb{Z}$. Now, $n+5$ is also an integer (found via a side calculation), hence the sum is also an integer. That is, $\frac{n^2+9n+20}{2} + n + 5 = \frac{n^2+9n+20}{2} + \frac{2n+10}{20} = \frac{n^2+11n+30}{2} = \frac{(n+1)^2+9(n+1)+20}{2} \in \mathbb{Z}$.

5. Solve the recurrence given by $a_0 = 2,\ a_1 = 3,\ a_n = -4a_{n-1} - 4a_{n-2}\ (n \geq 2)$.

The characteristic polynomial is $r^2 + 4r + 4 = (r + 2)^2$. Hence there is a double root of $r = -2$, and the general solution is $a_n = A(-2)^n + Bn(-2)^n$. We now use the initial conditions $2 = a_0 = A(-2)^0 + B \times 0 \times (-2)^0 = A$, and $3 = a_1 = A(-2)^1 + B \times 1 \times (-2)^1 = -2A - 2B$. We now solve the 2×2 linear system $\{2 = A, 3 = -2A - 2B\}$ to get $A = 2, B = -\frac{7}{2}$. Hence the specific solution is $a_n = 2(-2)^n - \frac{7}{2}n(-2)^n = -(-2)^{n+1} + 7n(-2)^{n-1}$.

WARNING: One must be careful with the laws of exponents to simplify at the end.

6. Let $a_n = n^{1.9} + n^{2.1}$. Prove or disprove that $a_n = O(n^2)$.

The statement is false. To disprove, let $M \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ be arbitrary. Choose $n = 1 + \max(n_0, \lceil M^{10} \rceil)$ [We need a specific choice of n, found via a side calculation.] Note that $n \in \mathbb{N}$ and $n \geq n_0$, so n is in the correct domain. Note also that $n > M^{10}$. Taking the tenth root gives $n^{0.1} > |M|$. Multiplying by n^2 gives $n^{2.1} > |M|n^2$. Now $|n^{1.9} + n^{2.1}| = n^{1.9} + n^{2.1} > n^{2.1} > |M|n^2 = |M||n^2| \geq M|n^2|$. Hence, $|n^{1.9} + n^{2.1}| > M|n^2|$.

7. Let F_n denote the Fibonacci numbers. Prove that $\forall n \in \mathbb{N}_0,\ F_{2n+1}^2 - F_{2n+2}F_{2n} = 1$.

We must use shifted induction, since the domain is $\mathbb{N}_0$. Strong induction is not needed, but you may use it if you like. Base case: $n = 0$, $F_0 = 0, F_1 = 1, F_2 = 1$, and we calculate $F_1^2 - F_2F_0 = 1^2 - 1 \times 0 = 1$.

Inductive case: Let $n \in \mathbb{N}_0$ be arbitrary, and assume that $F_{2n+1}^2 - F_{2n+2}F_{2n} = 1$. We now calculate with $x = F_{2n+3}^2 - F_{2n+4}F_{2n+2}$, trying to get to 1. We substitute $F_{2n+3} = F_{2n+2} + F_{2n+1}$ and $F_{2n+4} = F_{2n+3} + F_{2n+2}$, getting $x = (F_{2n+2} + F_{2n+1})^2 - (F_{2n+3} + F_{2n+2})F_{2n+2} = 2F_{2n+2}F_{2n+1} + F_{2n+1}^2 - F_{2n+3}F_{2n+2}$. We again substitute $F_{2n+3} = F_{2n+2} + F_{2n+1}$, getting $x = 2F_{2n+2}F_{2n+1} + F_{2n+1}^2 - (F_{2n+2} + F_{2n+1})F_{2n+2} = F_{2n+2}F_{2n+1} + F_{2n+1}^2 - F_{2n+2}^2 = F_{2n+2}(\underbrace{F_{2n+1} - F_{2n+2}}) + F_{2n+1}^2$. Lastly, we rearrange $F_{2n+2} = F_{2n+1} + F_{2n}$ into $-F_{2n} = F_{2n+1} - F_{2n+2}$, and substitute as marked, getting $x = F_{2n+2}(-F_{2n}) + F_{2n+1}^2$. Now, by the inductive hypothesis, $x = 1$, so $F_{2n+3}^2 - F_{2n+4}F_{2n+2} = x = 1$.

Note: This is a special case of Cassini's identity, as proved in the homework. However, you may not use results from homework problems on exams.